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Multiple bubbles in a Hele-Shaw cell

Giovani L. Vasconcelos*
Department of Mathematics, The Ohio State University, Columbus, Ohio, 43210
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A new class of exact solutions is reported for an infinite stream of identical groups of bubbles moving with a constant velocity U in a Hele-Shaw cell when surface tension is neglected. It is suggested that the existence of these solutions might explain some of the complex behavior observed in recent experiments on rising bubbles in a Hele-Shaw cell. Solutions for a finite number of bubbles in a channel are also obtained. In this case, it is shown that solutions with an arbitrary bubble velocity $U>V$, where V is the fluid velocity at infinity, can in general be obtained from a simple transformation of the solutions for $U=2V$.

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The original motivation for the research reported here was provided by the experimental work performed by Maxworthy [1] on bubbles rising in a tilted Hele-Shaw cell (a viscous fluid confined in a narrow gap between two glass plates). In Maxworthy's experiments, air was continuously injected at the bottom of cell leading to the formation of a stream of rising bubbles. He noticed, however, that when the spacing between bubbles was less than three or four bubble diameters, alternate bubbles would catch up with their upstream neighbors thus forming a pair of bubbles. Further pairing would then follow, thus creating stacks of four bubbles, eight bubbles, etc.

This complex bubble dynamics is reminiscent of similar behavior displayed by other nonlinear fluid systems, e.g., the "chaotic dripping faucet" [2]. In fact, recent experiments performed by Tritton and Egdell [3] have shown that bubbling from a submerged orifice does indeed exhibit a chaotic dynamics. In the case of bubbling into a tube [3] as well as dripping from a faucet [2], the chaotic behavior is determined by the complex dynamics governing the detachment of the bubble or drop from the orifice. In the case of a Hele-Shaw cell, on the other hand, the rich behavior seen in the experiments seems to stem from the hydrodynamic interaction between bubbles rather than the details of bubble injection. For instance, it will be argued below that the origin of the "pairing mechanism" might be related to the existence of steady solutions for multiple Hele-Shaw bubbles.

First I recall that the motion of viscous fluid in a Hele-Shaw cell is assumed to be governed by Darcy's law [4]:

$$\mathbf{v}=-\frac{b^2}{12\mu}[\nabla p-\rho\,\mathbf{g}], \tag{1}$$

where $\mathbf{v}$ is the fluid velocity (averaged across the gap), p is the pressure, b is the gap width, μ is the viscosity, ρ is the density, and $\mathbf{g}$ is the component of the gravitational acceleration parallel to the plates. Here I shall neglect surface tension effects so that the pressure is constant along the bubble surface. Note that in this two-dimensional model gravity plays no dynamical role when $\mathbf{g}$ is parallel to the cell centerline (as was the case in Maxworthy's experiment [5]) since it can be removed from the equations of motion by a suitable rescaling of variables [6]. The question then is how much of the complex behavior observed in the experiments can be explained on the basis of this simple model, without the need of introducing the complicating factor of surface tension, let alone three-dimensional thin film effects.

To the best of my knowledge, the first theoretical work to address this question is due to Burgess and Tanveer [7,8]. They found an exact solution for an infinite stream of identical bubbles in a channel when surface tension is neglected. Although the Burgess-Tanveer solution is relevant to the stream of approximately identical bubbles observed in the experiments [1], it cannot account for the subsequent "pairing" that occurs once the initial stream of bubbles becomes unstable.

In this paper I report a rather general class of exact solutions for steady bubbles in a rectilinear Hele-Shaw cell when surface tension is neglected. These solutions correspond to an infinite stream of identical *groups* of bubbles in a channel, in which the group centers are at a distance $2L$ apart from each other. In other words, the solutions are periodic in the x direction (taken to be along the channel centerline) with period $2L$. Let me however emphasize that the solutions presented below are general enough to allow any number m of bubbles per unit cell. (Of course, the solutions with $m=1$ recovers the Burgess-Tanveer solutions.)

It should be noted at this stage that in view of the simplifications of the model, chiefly among them the absence of surface tension, the solutions above cannot be compared directly with the experiments. Nonetheless, the existence of this class of exact solutions might provide a theoretical framework within which one can understand (at least qualitatively) the pairing phenomena observed in the experiments [1]. Accordingly, the solutions for higher m (or rather their counterparts with small surface tension) might act as an "attractor" for the dynamics once the initial stream of bubbles (or a solution with smaller m) becomes unstable. In this fashion, a succession of instabilities ("bifurcations") could then occur as the flow rate is increased, thus leading to successive pairing ("period doubling"). Clearly a more complete study of this complex process must necessarily take into account surface tension effects. Such a task, however, is beyond the scope of the present paper.

*Present address: Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany.

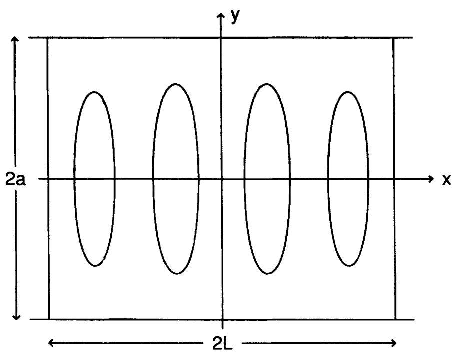

FIG. 1. The unit cell with four bubbles.

After this introductory discussion we shall now turn to the description of our solutions. For definiteness, we assume that gravity plays no role, i.e., the cell, whose width is $2a$, is horizontally placed. We then consider the problem of a periodic array of m bubbles per unit cell all traveling with speed U along the channel centerline (assumed to be the x axis). The fluid inside the bubbles has a negligible viscosity and is kept at constant pressure. In the plane $z=x+iy$ moving with the bubbles, the flow is described by the complex potential $W(z)=\phi(x,y)+i\psi(x,y)$, where ϕ is the velocity potential in the moving frame [i.e., $\phi=-(b^2/12\mu)p-Ux$] and ψ is the stream function. Now let $\mathscr{C}_k$ denote the fluid-bubble interface corresponding to the kth bubble. Then the function $W(z)$ must be analytic in the fluid region and satisfy the following boundary conditions:

$$W=-Ux+\phi_k \text{ on } \mathscr{C}_k, \quad k=1,2,\ldots,m, \tag{2}$$

where the ϕ_k's are real constants. Equation (2) encodes in a single expression both the fact that the interfaces must be streamlines of the flow [i.e., the imaginary part of W is constant (zero) on $\mathscr{C}_k$] and the condition that the fluid pressure p is constant along these interfaces (i.e., the real part of the complex potential in the lab frame is constant on $\mathscr{C}_k$). (For a more detailed discussion of these boundary conditions see, e.g., Ref. [9].) We also have

$$\operatorname{Im} W=\pm(V-U)a \text{ at } y=\pm a, \tag{3}$$

since the solid walls must be streamlines of the flow. Here V is the average fluid velocity across the channel in the x direction [7], and without loss of generality we set $V=1$.

We assume here that the bubbles are symmetrical with respect to the channel centerline (the x axis) and suppose furthermore that the flow is also symmetrical about the y axis; see Fig. 1. As a result, we can reduce the problem to a domain corresponding to one quarter of the original unit cell (the upper right quarter say). This is advantageous because now our domain of interest is simply connected. Next we consider the conformal mapping $z=f(\zeta)$ that maps the interior of the unit semicircle in the ζ plane onto our fluid domain; see Fig. 2. Although most of the formalism that follows applies (with minor differences [10]) to cases with either an even number or an odd number of bubbles per unit cell, we shall for simplicity focus on the former, that is, we take $m=2N$.

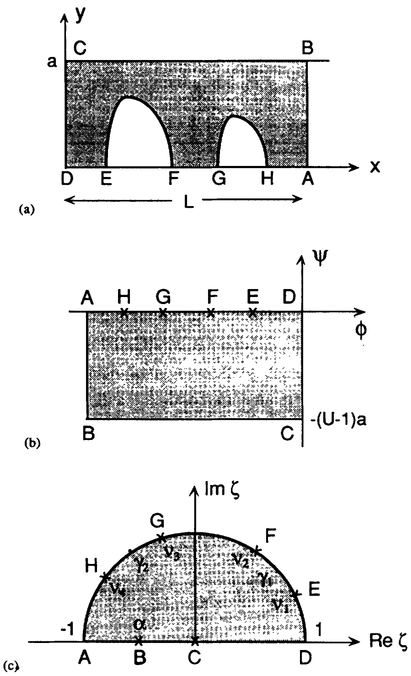

FIG. 2. The flow domain corresponding to Fig. 1: (a) the z plane, (b) the W plane, and (c) the ζ plane.

Defining $\Phi(\zeta)=W(f(\zeta))$, we can then view the complex potential as a conformal mapping $W=\Phi(\zeta)$ from the ζ unit semicircle to the W plane (see Fig. 2). Because the flow domain in the W plane is the interior of a polygon (rectangle), one can easily construct the appropriate conformal mapping. One then finds that $\Phi(\zeta)$ is determined by

$$\frac{d\Phi}{d\zeta}=\frac{iC}{[\zeta(\zeta-\alpha)(1-\alpha\zeta)]^{1/2}}, \tag{4}$$

where α, and C are real parameters taking values in the ranges: $-1<\alpha<0$ and $C>0$.

To obtain the mapping function $f(\zeta)$ we shall make use of a theorem due to Tian and Vasconcelos [11]. Their result implies, in particular, that if a curve $\mathscr{C}$ is a solution for a Hele-Shaw bubble moving with speed U in a channel (in the absence of surface tension), then the curve $\tilde{\mathscr{C}}$ obtained from a 90° rotation of $\mathscr{C}$ is also a solution moving with the same speed U. In our conformal-mapping representation, the new (rotated) solution reads as $z=g(\zeta)=-if(\zeta)$, with the corresponding complex potential $W=\Sigma(\zeta)$ being given by [11]

$$\Sigma(\zeta)=i[\Phi(\zeta)+Uf(\zeta)]. \tag{5}$$

Solving for $f(\zeta)$ then yields

$$f(\zeta)=-\frac{1}{U}[\Phi(\zeta)+i\Sigma(\zeta)]. \tag{6}$$

The advantage of this formulation is that the mapping function $\Sigma(\zeta)$ can now be easily computed since the flow domain in the W plane for the rotated problem is also the interior of a polygon. One then finds that $\Sigma(\zeta)$ is determined by [10]

$$\frac{d\Sigma}{d\zeta}=\frac{K\prod_{i=1}^{N}(1-2\cos\gamma_i\zeta+\zeta^2)}{\left[\zeta(\zeta-\alpha)(1-\alpha\zeta)\prod_{i=1}^{2N}(1-2\cos\nu_i\zeta+\zeta^2)\right]^{1/2}}, \quad (7)$$

where K, γ_i, and ν_i are real parameters taking values in the ranges: $K>0$, $0<\nu_1<\nu_{2N}<\pi$, and $\nu_{2i-1}<\gamma_i<\nu_{2i}$, for $i=1,2,\ldots,N$.

Thus, Eqs. (4), (6), and (7) give the generic solution to the problem. This solution must, in addition, satisfy specific constraints as we shall now discuss. For instance, if we denote by $[W]_{CD}$ the jump in the complex potential as we go from point C to point D in Fig. 2, that is, $[W]_{CD}=W(0)-W(ia)$, then the requirement that $[W]_{CD}=i(U-1)a$ yields the following condition:

$$[\Phi]_{CD}=i(U-1)a. \quad (8)$$

Similarly, the conditions $[f]_{DC}=ia$ and $[f]_{CB}=L$ imply, respectively,

$$[\Sigma]_{CD}=a,$$
$$[\Phi]_{BC}+i[\Sigma]_{BC}=UL. \quad (9)$$

Finally, one must also impose the conditions

$$\mathrm{Im}\left[\int_{e^{i\nu_{2k-1}}}^{e^{i\nu_{2k}}} f'(\zeta)d\zeta\right]=0, \quad k=1,2,\ldots,N, \quad (10)$$

where the prime denotes derivative and the contour integrals are along the corresponding arcs on the unit circle: $\zeta=e^{i\theta}$, $\nu_{2k-1}<\theta<\nu_{2k}$. The conditions (10) enforce the symmetry about the channel centerline.

Thus for given U and L there are $3N+3$ unknowns $(C,K,\alpha;\gamma_i,\nu_{2i-1},\nu_{2i}$, with $i=1,\ldots,N)$ and $N+3$ conditions [Eqs. (8)–(10)], so that the conformal map $f(\zeta)$ possess then $2N$ free parameters. These correspond, of course, to the $2N$ geometrical parameters of the solutions: the area and the position (along the x axis) of each of the N bubbles.

Now we turn to discuss in some detail the particular case of exact solutions for a *finite* number of bubbles in unbounded (not periodic) domains. Note that in this case the velocity V introduced in Eq. (3) represents the fluid velocity at infinity.

One such solution can be easily obtained from our general solution by simply taking $L\to\infty$, which corresponds to letting $\alpha\to-1$ (see Fig. 2). Thus, setting $\alpha=-1$ in the preceding equations yields a (2N+1)-parameter family of solutions for a group of $2N$ bubbles moving with velocity U along the centerline of the channel. Another solution can also be obtained by considering now L as a fixed parameter and taking instead $a\to\infty$. This corresponds to letting $\alpha\to0$ (see Fig. 2). Thus taking $\alpha=0$ in the equations above and considering the "rotated" conformal mapping $z=g(\zeta)$ gives a $(2N+1)$-parameter family of solutions for $2N$ bubbles in a channel of width $2L$. Note that in this case, however, the bubbles are aligned *perpendicularly* to the channel. Since the channel centerline is a streamline of the flow, one can alternatively think of these solutions as giving a group of N bubbles moving "side by side" in a channel of width L [10].

In the case of unbounded domains, the solutions with $U=2$ possess an interesting property that deserves further discussion. Let us then indicate with a zero subscript the corresponding conformal mappings for this case, that is, we write $z=f_0(\zeta)$ and

$$f_0(\zeta)=-\frac{1}{2}[\Phi_0(\zeta)+i\Sigma_0(\zeta)]. \quad (11)$$

One can now easily convince oneself that the mapping function $\Phi(\zeta)$ for an arbitrary velocity $U>1$ is simply given by $\Phi(\zeta)=(U-1)\Phi_0(\zeta)$. The conformal mapping $z=f(\zeta)$ for the general case can thus be written as

$$f(\zeta)=-\frac{1}{U}[(U-1)\Phi_0(\zeta)+i\Sigma_0(\zeta)]. \quad (12)$$

Defining $\mu=1-2/U$, we can recast this in the form

$$f(\zeta)=-\frac{1}{2}[(1+\mu)\Phi_0(\zeta)+i(1-\mu)\Sigma_0(\zeta)]. \quad (13)$$

Now note that the shapes of the bubbles moving with velocity $U>1$ are given by the parametric equations

$$x^k(\theta)+iy^k(\theta)=f(e^{i\theta}),$$
$$\nu_{2k-1}\leqslant\theta\leqslant\nu_{2k}, \quad k=1,\ldots,N, \quad (14)$$

which in view of Eqs. (11) and (13) can be written as

$$x^k(\theta)=(1+\mu)x_0^k(\theta),$$
$$y^k(\theta)=(1-\mu)y_0^k(\theta), \quad (15)$$

where $x_0^k(\theta)+iy_0^k(\theta)=f_0(e^{i\theta})$ are the parametric equations for the bubble shapes with $U=2$. Thus, we see that solutions for any value of $U>1$ can be obtained from the solutions with $U=2$ by a mere rescaling of the spatial coordinates. This was first noticed by Millar [12] in the context of the Taylor-Saffman [13] solutions for a single bubble in the channel geometry. We have shown here that this result holds in general for an arbitrary number of bubbles in an unbounded (rectilinear) geometry.

It is perhaps worth mentioning that there seems to be a connection between the "special nature" of the $U=2$ solutions (in the sense described above) and the so-called "selection problem" for Saffman-Taylor fingers. The latter refers to the fact that experimentally only fingers with width $\lambda=1/2$ (corresponding to $U=2$) are observed in the limit of vanishing surface tension. (This problem is now fairly well understood theoretically: asymptotics beyond all orders [14] predict that for a given value of surface tension λ has a discrete set of values, all of which converge to $\lambda=1/2$ as the surface tension approaches zero.)

Such an analogy suggests, for instance, the interesting possibility that in the case of periodic solutions, for which no "special" value of U exists, the zero surface-tension limit may not give a unique value for the bubble speed U (for fixed bubble area). Indeed, in the numerical solutions obtained by Burgess and Tanveer [7] for a stream of bubbles with surface tension, there was no clear indication that the different branches of solutions were converging to the same value of U as the surface tension parameter was decreased. I am, however, unaware of any quantitative experimental study of a stream of bubbles in a Hele-Shaw cell against which to test the prediction above. Clearly further work is required to settle this question.

As a final comment, I would like to add that the general solution with an odd number $(m=2N-1)$ of bubbles per unit cell can be easily obtained by setting $\nu_2=-\nu_1$ and $\gamma_1=0$ in Eq. (7), and then making appropriate changes in the subsequent formulas, the details of which will be worked out in a future publication [10].

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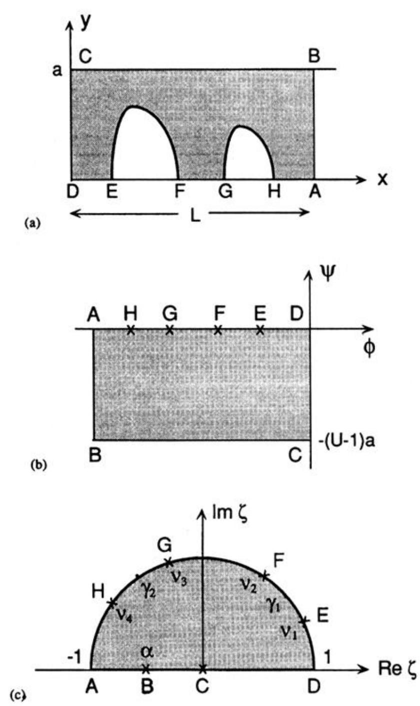

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